ISRAEL JOURNAL OF MATHEMATICS, Vol. 40, Nos. 3–4, 1981

EXTREMELY WEAKLY UNCONDITIONALLY CONVERGENT SERIES

BY
JOHN ELTON†

ABSTRACT

It is proved that if $\Sigma_{i=1}^{\infty} x_i$ is a non-convergent series in a Banach space X such that $\Sigma_{i=1}^{\infty} |f(x_i)| < \infty$ for all extreme points f of the unit ball of X^*, then X contains a subspace isomorphic to c_0, improving a result of Bessaga and Pelczynski. The proof uses Fonf's result that Lindenstrauss–Phelps spaces contain isomorphs of c_0.

We give a simple proof of

THEOREM 1. *Let (x_i) be a normalized basis for a Banach space X, and suppose $\Sigma_{i=1}^{\infty} |f(x_i)| < \infty$ for all $f \in \operatorname{ext} B^*$, the set of extreme points of the unit ball B^* of X^*. Then every subspace of X has a subspace isomorphic to c_0.*

As a consequence, we obtain the following improvement of a result of Bessaga and Pelczynski [1]:

COROLLARY. *A Banach space X has a subspace isomorphic to c_0 if and only if there is a sequence (x_i) in X so that $\Sigma_{i=1}^{\infty} |f(x_i)| < \infty$ for every $f \in \operatorname{ext} B^*$ but $\Sigma_{i=1}^{\infty} x_i$ fails to converge.*

This also generalizes the result of V. Fonf [3] that Lindenstrauss–Phelps spaces (infinite-dimensional real Banach spaces whose dual balls have countably many extreme points) contain isomorphs of c_0 hereditarily. The proof of Theorem 1 uses the following strengthening of a result of Kadets and Fonf [4]:

THEOREM 2. *Suppose that X is an infinite-dimensional real Banach space for which $\operatorname{ext} B^*$ can be covered by a countable union of norm-compact sets. Then X is isomorphic to a Lindenstrauss–Phelps space.*

† Supported in part by NSF-MCS-8002393.
Received May 3, 1981

Kadets and Fonf showed that if X satisfies the hypothesis of Theorem 2, then X is not reflexive. Our proof of Theorem 2 is obtained by only a slight modification of the beginning of Fonf's proof in [3].

PROOF OF THEOREM 2. Let $\operatorname{ext} B^* \subset \bigcup_{n=1}^{\infty} k_n$, where each k_n is compact in the norm topology of X^*. We may assume $k_n \subset B^*$ for each n. Let $\varepsilon_n \downarrow 0$, $\varepsilon_1 < 1$. Let $\mathcal{F}_n$ be a finite $\varepsilon_n/2$ net for k_n. Define a new norm on X by the formula

$$|||x||| = \sup \bigcup_{n=1}^{\infty} \{(1+\varepsilon_n)|f(x)| : f \in \mathcal{F}_n\}.$$

Let V^* be the unit ball of $(X^*, |||\cdot|||)$. It is easy to see that $\|x\| \leqq |||x||| \leqq (1+\varepsilon_1)\|x\|$, so the new norm is equivalent to the old, and $V^* \supset B^*$. We shall show that $\mathring{V}^* \supset B^*$, where $\mathring{V}^* = \{f \in X^* : |||f||| < 1\}$. If not, there exists $f_0 \in X^*$ with $|||f_0||| = \|f_0\| = 1$. Let $F_0 \in X^{**}$ such that $F_0(f_0) = 1$, $|||F_0||| = 1$, so $\|F_0\| = 1$ also. It was shown in [4] that X^* is separable. Thus there is $f \in \operatorname{ext} B^*$ such that $F_0(f) = 1$ by a theorem of Bessaga and Pelczynski [2]. Now $f \in k_n$ for some n, so $\|f - g\| < \varepsilon_n/2$ for some $g \in \mathcal{F}_n$. Thus $F_0(g) > 1 - \varepsilon_n/2$, so $F_0((1+\varepsilon_n)g) > (1+\varepsilon_n)(1-\varepsilon_n/2) > 1$, a contradiction to $|||F_0||| = 1$ since $(1+\varepsilon_n)g \in V^*$. Thus the inclusion holds. From Milman's theorem, $\operatorname{ext} V^* \subset$ w*-closure $\bigcup_{n=1}^{\infty}\{\pm(1+\varepsilon_n)f : f \in \mathcal{F}_n\}$; but any w*-limit point of the set $\bigcup_{n=1}^{\infty}\{\pm(1+\varepsilon_n)f : f \in \mathcal{F}_n\}$ lies in $B^* \subset \mathring{V}^*$, so could not be an extreme point of V^*. Thus $\operatorname{ext} V^* \subset \bigcup_{n=1}^{\infty}\{\pm(1+\varepsilon_n)f : f \in \mathcal{F}_n\}$ which proves the theorem.

LEMMA. *Let X satisfy the hypothesis of Theorem* 1. *Then (x_i) is a shrinking basis for X.*

PROOF. It is sufficient to show that if (u_i) is a normalized block basis on (x_i), then $u_i \to 0$ weakly. But $\sum_{i=1}^{\infty}|f(x_i)| < \infty$ for $f \in \operatorname{ext} B^* \Rightarrow f(u_i) \to 0$ for $f \in \operatorname{ext} B^*$, since each u_i has coefficients in its expansion according to the basis (x_i) which are bounded. Thus the conclusion follows from Rainwater's theorem [7].

PROOF OF THEOREM 1. Without loss of generality we may assume X is a real space. Let (f_i) be biorthogonal to (x_i). Define $T : l_1 \to X^*$ by $T(\sum_{i=1}^{\infty} \alpha_i e_i) = \sum_{i=1} \alpha_i f_i$, where (e_i) is the unit vector basis of l_1. If $T \mid [e_i]_{i=N}^{\infty}$ is an isomorphism for some N, then it is easily seen that $(x_i)_{i=N}^{\infty}$ (and hence $(x_i)_{i=1}^{\infty}$) is equivalent to the unit vector basis of c_0. So suppose this fails for every N. Then by a standard argument we may construct a normalized block basis (u_j) of (e_i) such that $\|Tu_j\| < 2^{-j}$, so that $T \mid [u_j]_{j=1}^{\infty}$ is a compact operator (this argument is due to Kato [5]). Write

$$u_j = \sum_{i=m_j+1}^{m_{j+1}} \lambda_i e_i, \qquad \sum_{i=m_j+1}^{m_{j+1}} |\lambda_i| = 1, \qquad m_1 < m_2 < \cdots.$$

Let

$$h_j = Tu_j = \sum_{i=m_j+1}^{m_{j+1}} \lambda_i f_i, \quad y_j = \sum_{i=m_j+1}^{m_{j+1}} (\operatorname{sgn} \lambda_i) x_i, \quad Y = [y_i]_{j=1}^{\infty}.$$

Then (h_j) is biorthogonal to (y_j) and so by the lemma, $(\tilde{h}_j)$ is a basis for Y^* where $\tilde{h}_j = h_j \mid Y$. Let $k_n = \{g \in B^* : \Sigma_{i=1}^{\infty} |g(x_i)| \leqq n\}$, $n = 1, 2, \cdots$. Thus $\operatorname{ext} B^* \subset \bigcup_{n=1}^{\infty} k_n$. If $g \in k_n$, $\Sigma_{j=1}^{\infty} |g(y_j)| \leqq \Sigma_{i=1}^{\infty} |g(x_i)| \leqq n$ (even though $\|y_j\| > 2^j$). So if $g \in k_n$ and $\tilde{g} = g \mid Y = \Sigma_{j=1}^{\infty} \beta_j \tilde{h}_j$, then $\Sigma_{j=1}^{\infty} |\beta_j| = \Sigma_{j=1}^{\infty} |g(y_j)| \leqq n$. Let

$$C_n = \left\{ \sum_{j=1}^{\infty} \beta_j \tilde{h}_j : \sum_{j=1}^{\infty} |\beta_j| \leqq n \right\} = nT(\operatorname{Ball}[u_j]_{j=1}^{\infty}) \mid Y.$$

This is norm-compact in Y^*. Now

$$\operatorname{ext}(\operatorname{Ball} Y^*) \subset \{g \mid Y : g \in \operatorname{ext} B^*\} \subset \bigcup_{n=1}^{\infty} \{g \mid Y : g \in k_n\} \subset \bigcup_{n=1}^{\infty} C_n.$$

Thus Y is isomorphic to a Lindenstrauss–Phelps space by Theorem 2, so Y contains an isomorph of c_0 by Fonf's result [3]. By a standard argument, any subspace of X contains a normalized basic sequence whose closed linear span satisfies the hypothesis of Theorem 1, so X actually contains isomorphs of c_0 hereditarily.

This completes the proof of Theorem 1.

The corollary follows from Theorem 1 by a standard argument.

REMARKS. (1) R. Haydon and E. Odell proved the conclusion of Theorem 1 holds under the stronger hypothesis $\Sigma_{i=1}^{\infty} |f(x_i)| < \infty$ for all $f \in$ w*-closure ext B^*. However, there exist simple examples for which $\Sigma_{i=1}^{\infty} |g(x_i)| < \infty$ for all $g \in \operatorname{ext} B^*$ but not for all $g \in$ w*-closure ext B^*.

(2) We do not know of any example of a Banach space satisfying the hypothesis of Theorem 1 which is not isomorphic to a Lindenstrauss–Phelps space. In fact, we do not know of any space having separable dual which contains isomorphs of c_0 hereditarily which is not isomorphic to a Lindenstrauss–Phelps space.

We would like to thank P. Wojtaszczyk for pointing out the work of Fonf and of Kadets and Fonf.

REFERENCES

1. C. Bessaga and A. Pelczynski, *On bases and unconditional convergence of series in Banach spaces*, Studia Math. **17** (1958), 151–164.

2. C. Bessaga and A. Pelczynski, *On extreme points in separable conjugate spaces*, Israel J. Math. **4** (1966), 262–264.

3. V. Fonf, *One property of Lindenstrauss–Phelps Spaces*, Functional Anal. Appl. **13** (1) (1979), 66–67.

4. M. Kadets and V. Fonf, *Some properties of the set of extreme points of the unit ball of a Banach space*, Math. Notes **20** (3) (1976), 737–739.

5. T. Kato, *Perturbation theory for nullity deficiency and other quantities of linear operators*, J. Analyse Math. **6** (1958), 273–322.

6. J. Lindenstrauss and R. Phelps, *Extreme point properties of convex bodies in Banach spaces*, Israel J. Math. **6** (1968), 39–48.

7. J. Rainwater, *Weak convergence of bounded sequences*, Proc. Amer. Math. Soc. **14** (1963), 999.

DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF TEXAS
AUSTIN, TX 78712 USA